Estimation of shear wave velocity in a compressible Gibson half-space by inverting Rayleigh wave phase velocity

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SUMMARY

Shear wave velocities in a compressible Gibson half-space (a non-layered earth model) are estimated by inverting Rayleigh wave phase velocity. An analytical dispersion law of Rayleigh-type waves in a compressible Gibson half-space is in an algebraic form (Vardoulakis and Verttos, 1988), which makes our inversion processing extremely simple and fast. The convergence of the weighted damping solution is guaranteed through selection of the damping factor using the Levenberg-Marquardt method (L-M) (Marquardt, 1963). Calculation efficiency is achieved by reconstructing a weighted damping solution using the singular value decomposition (SVD) techniques (Golub and Reinsch, 1970). One real example is presented and verified by borehole S-wave velocity measurements. Results of this example are also compared with results of the layered-earth model (Xia et al., in review).

INTRODUCTION

The shear (S)-wave velocities of near-surface materials (such as soils) are of fundamental interest in many engineering studies. There are three basic ways to perform shear tests in-situ in soil mechanics: in-situ shear box, shear vane, and penetration (e.g., Whitlow, 1995). During these tests, either a soil sample needed to be carefully cut or remolded (the first method) or invasive penetrations need to be performed (the second and third methods). Analysis of surface wave dispersion offers a rapid, non-invasive, and cost-effective way to evaluate shear strength.

Ground roll is a Rayleigh-type surface wave that travels along or near the surface of the ground. It is usually characterized by relatively low velocity, low frequency, and high amplitude (Sheriff, 1991). Stokoe and Nazarian (1983) and Nazarian et al. (1983) presented a surface-wave method, called Spectral Analysis of Surface Waves (SASW), that analyzes the ground roll generated from impact acoustic sources to produce near-surface S-wave velocity profiles. SASW has attracted the attention of engineering scientists and has been widely applied to many engineering projects (e.g., Sanchez-Salinero et al., 1987; Sheu et al., 1988; Stokoe et al., 1989; Song et al., 1989; Gucunski and Woods, 1991; Hiltunen, 1991; Stokoe et al., 1994). All of this research is focused on a layered-earth model.

The Kansas Geological Survey conducted a research program to estimate S-wave velocity from ground roll. The program consists of three phases: acquisition of wide band ground roll, creation of efficient and accurate algorithms organized in a basic data processing sequence designed to extract Rayleigh wave dispersion curves from ground roll, and development of stable and efficient inversion algorithms to obtain S-wave velocity profiles. Park et al. (1996a) introduced a method, Multi-channel Analysis of Surface Waves using Vibroseis (MASWV), that can produce wide band ground roll. Park et al. (1996b) presented an unique technique, Cross-Correlation of Stacked Amplitudes with Sweep (CCSAS), that can efficiently extract accurate Rayleigh wave phase velocities from ground roll.

For a non-layered medium surface waves are always dispersive due to inhomogeneity of the elastic properties in real soil bodies (e. g., Richart et al., 1970). Since in a half-space of sedimentary granular soil under geostatic state of initial stress, the density and the Poisson's ratio do not vary considerably with depth. In such an earth body, the dynamic shear modulus is the parameter which mainly affects dispersion of propagating wave (Vardoulakis and Verttos, 1988). Gibson (1967) introduced an assumption of shear modulus distribution linearly with depth in an inhomogenous elastic half-space, which is called a compressible Gibson half-space. Vardoulakis and Verttos (1988) derived an analytical dispersion law of Rayleigh-type waves in the compressible Gibson half-space. The dispersion law is in an algebraic form, which makes our inversion processing extremely simple and fast, Based on the dispersion law, an iterative solution technique to a weighted damping equation using the L-M method and the SVD proved very effective. Reconstruction of shear wave velocities by inversion of Rayleigh wave phase velocity could be achieved in real time in situ.

THE METHOD

In this section, we first list the dispersion law of the Gibson half-space and the partial derivatives of the dispersion law, then formulate an inversion algorithm by the L-M method and the SVD technique.

Gibson half-space and dispersion law The Gibson half-space (Gibson, 1967) is defined as an inhomogeneous elastic half-space $z \leq 0$ with a constant density ρ and Poisson's ratio V and a dynamic shear modulus G increasing linearly with depth. The shear modulus variation in a Gibson half-space is given by:

$$G = G_0(1 + mz), \tag{1}$$

where $G_0 > 0$ is the shear modulus at the free surface, m is a measure of inhomogeneity and possesses the dimension of inverse length. The limiting value $m = 0$ corresponds to the homogeneous elastic half-space, where Rayleigh waves do not exhibit dispersion.

Vardoulakis and Verttos (1988) derived an approximate algebraic form of the dispersion law for the fundamental mode of the Rayleigh wave in the Gibson half-space (Eq. 1).

$$C \cong \frac{1}{\Omega_\nu} + \sqrt{\frac{1}{\Omega_\nu^2} + \frac{1}{0.35(3.6-\nu)}}, \qquad 0.25 \le \nu \le 0.5 \tag{2}$$

where

$$C = c / v_{s0}, \tag{3}$$

$$\Omega_\nu = 0.56(3.6-\nu)\Omega/(1.5+\nu), \tag{4}$$

c is the **Rayleigh** wave phase velocity, v_{s0} is the shear wave velocity at the free surface,

$$v_{s0} = \sqrt{G_0/\rho}, \tag{5}$$

$$\Omega = 2\pi f / m v_{s0}, \text{ and} \tag{6}$$

f is frequency in Hz. C and Ω are called dimensionless velocity and dimensionless frequency, respectively. Vardoulakis and Verttos (1988) stated that the relative error induced by the approximation (Eq. 2) is l-3 percent.

Based on Eq. 1, the shear wave velocity at depth z can be written as

$$v_s(z) = \sqrt{G/\rho} = \sqrt{G_0(1+mz)/\rho}. \tag{7}$$

Similarly, Young's modulus, E, and the bulk modulus, k, at depth z can be related to the shear modulus and Poisson's ratio $E(z) = 2(1+\nu)G$ and $k(z) = 2(1+\nu)G/3(1-2\nu)$, respectively.

Partial derivatives of disnersion law Eq. 2 shows that the fundamental mode of the Rayleigh wave in the Gibson half-space, c, is a function of shear wave velocity at surface v_{s0}, the measure of inhomogeneity m, and Poisson's ratio V. Because m always occurs in a form of mv_{s0}, it is easy to calculate derivatives if defining a new variable as $d = mv_{s0}$. *The* partial derivatives of Eq. 2 are listed as follows.

$$\frac{\partial c}{\partial v_{s0}} = \frac{1}{\Omega_\nu} + \sqrt{\frac{1}{\Omega_\nu^2} \cdot \frac{1}{0.35(3.6-\nu)}}. \tag{8}$$

$$\frac{\partial c}{\partial d} = \frac{1}{m\Omega_\nu^2}\left[\Omega_\nu + \left(\frac{1}{\Omega_\nu^2} + \frac{1}{0.35(3.6-\nu)}\right)^{-1/2}\right], \text{ and} \tag{9}$$

$$\frac{\partial c}{\partial \nu} = -\frac{1}{\Omega_\nu^2}\Omega_\nu' + 0.5\left(\frac{1}{\Omega_\nu^2} + \frac{1}{0.35(3.6-\nu)}\right)^{-1/2}\left[\frac{-2\Omega_\nu'}{\Omega_\nu^3} + \frac{1}{0.35(3.6-\nu)^2}\right], \tag{10}$$

where

$$\Omega_\nu' = -\frac{0.56}{(1.5+\nu)}\left(1 + \frac{3.6-\nu}{1.5+\nu}\right)\Omega \tag{11}$$

is the derivative of Ω with respect to Poisson's ratio V. As indicated by Eq. 2, Poisson's ratio is in the range of 0.25 to 0.5. This constraint is replaced by following variable substitutions.

$$V = 0.5(b_2 + b_1) + 0.5(b_2 - b_1)\sin x, \quad 0.25 \le b_1 \le b_2 \le 0.5, \text{ and } -\infty < x < \infty. \tag{12}$$

In this case, Eq. 10 will be replaced by the following equation.

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \nu}\frac{\partial \nu}{\partial x} = 0.5(b_2 - b_1)\cos x \,\frac{\partial c}{\partial \nu}. \tag{10'}$$

Inversion algorithm By reforming variables of Eq. 2, Eqs. 8, 9, and 10' show that the fundamental mode of the Rayleigh wave in the Gibson half-space, c, is a function of v_{s0}, $d = mv_{s0}$ and x, which is defined by Eq. 12. Earth model parameters can be represented as the elements of a vector $\vec{x}$ of length 3, $\vec{x} = [v_{s0},\ d, x]^T$. Similarly, the measurements (data) of Rayleigh wave phase velocities at m different frequencies can be represented as the elements of a vector $\vec{b}$ of length m, $\vec{b} = [b_1, b_2, b_3, ..., b_m]^T$ (the subscript m should not be confused with m, the measure of inhomogeneity).

We defined the objective function as

$$\Phi = \|\mathbf{J}\vec{x} - \vec{b}\|_2 \mathbf{W} \|\mathbf{J}\vec{x} - \vec{b}\|_2 + \lambda \|\vec{x}\|_2^2, \tag{13}$$

where a is a damping factor, W is a weighting matrix, $\vec{b} - \vec{c}(\vec{x}_0) = \vec{b}$, $\vec{x}_0$ is an initial estimation of earth model parameters for the Gibson half-space, and J is the Jacobi matrix consisting of m rows and 3 columns. The elements of the Jacobi matrix are the first order partial derivatives of $\vec{c}$ with respect to earth model parameters and defined by Eqs. 8, 9, and 10'.

Phase velocities at different frequencies have different resolving powers to earth model parameters. This property is used to determine the weighting matrix W, which is a diagonal matrix with w_i being the ith element on the diagonal line of the matrix. Basically, w_i is defined by a difference of measurements with respect to frequencies. Since the number of data points contained in the dispersion curve is generally more than the number of parameters used to model the subsurface, Eq. 13 is usually solved by optimization techniques. The damping least-squares solution to Eq. 13 can find from the equation, $(\mathbf{A}^T\mathbf{A}+\lambda\mathbf{I})\vec{x} = \mathbf{A}^T\vec{d}$. After applying the SVD technique to the equation, we obtain a solution as follows.

$$\vec{x} = \mathbf{V}(\Lambda^2+\lambda\mathbf{I})^{-1}\Lambda\,\mathbf{U}^T\vec{d}\,. \tag{14}$$

where with $V = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots \vec{v}_n]$ where $\vec{v}_i$ (i = 1, 2, ... n, n is equal to 3 in our inverse problem) are eigenvectors of matrix $\mathbf{A}^T\mathbf{A}$, $\Lambda = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots \sqrt{\lambda_n})$. This diagonal matrix has elements that are the square root of the eigenvalues of $\mathbf{A}^T\mathbf{A}$, $\mathbf{U} = [\vec{u}_1, \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_n]$, $\vec{u}_i = \mathbf{A}\vec{v}_i/\lambda_i$ (i = 1, 2, . . . , n), $\vec{d} = \mathbf{L}\vec{b}$, $W = \mathbf{L}^T\mathbf{L}$, and I is the unit matrix. Eq. 14 is dependent on the damping factor. Marquardt (1963) pointed out that the damping factor controls the direction of $\vec{x}$ and speed of convergence. Eq. 15 is an interpolation between the least-squares solution and the steepest descent solution. Several different values of λ need to be tried to insure a proper damping factor is found. Directly inverting the matrix $(\mathbf{A}^T\mathbf{A}+\lambda\mathbf{I})$ is not an efficient way to search for the proper damping factor. The inverse matrix of $(\mathbf{A}^T\mathbf{A}+\lambda\mathbf{I})$ must be calculated every time when the damping factor λ is changed. Because the inverse matrix in Eq. 15 is a diagonal matrix, inverting this matrix requires simple division. Based on Eq. 14, to find a proper modification of $\vec{x}$, only matrix multiplication is needed for the new damping factor.

A REAL WORLD EXAMPLE

A test conducted during the Winter of 1995 included collection of surface wave data near the Kansas Geological Survey in Lawrence, Kansas, using the MASWV acquisition method (Park et al., 1996a). An IVI MiniVib was used as the energy source. Forty groups of 10 Hz geophones were spaced 1 m apart with the first group of geophones two meters away from a test well. The source was located adjacent to the geophone line relative to the test well with a nearest source offset of 27 m. A linear up-sweep with frequencies ranging from 10 to 200 Hz and lasting 10 seconds was generated for each shot station. At the same time, three-component borehole data were acquired to obtain P-wave and S-wave velocity vertical profiles. Fig. la shows the dispersion curve of Rayleigh wave phase velocities for frequencies ranging from 15 to 80 Hz using CCSAS processing techniques (Park et al., 1996b) and modeled Rayleigh wave phase velocities.

The final inverted model was obtained after 22 iterations requiring in less than three seconds on a PC with an Intel Pentium 133 MHz processor. The root-means-square error between measured phase velocities and modeled phase velocities is reduced from 96 m/s to 28 m/s. The inverted shear wave velocity at surface, the measure of inhomogeneity, and Poisson's ratio are 104.3 m/s, 2.76 l/m, and 0.369, respectively. The S-wave velocity profiles are shown in Fig. lb, where the density is assumed as 2.0 g/cm^3. The average relative error is about 15% compared with borehole results. P-wave velocities are also calculated using these results and found the average relative error to be about 20% compared with borehole results.

DISCUSSION AND CONCLUSIONS

Poisson's ratio is a very active variable in three earth model parameters. For a given frequency and degree of inhomogeneity, a higher Poisson's ratio cause a higher phase velocity. To obtain reasonable results of S-wave velocities and achieve higher convergence speed it is necessary to impose a proper constraint on Poisson's ratio.

We have presented iterative solutions to the weighted damping equation by the L-M method and the SVD techniques. The real example demonstrated that inverting Rayleigh wave dispersion data can provide reliable S-wave velocities, which are verified by borehole S-wave velocity measurements. The real example also demonstrated calculation efficiency and stability of the inverse procedure. This report only shows preliminary results of our research program. More field tests are needed to evaluate this approach.

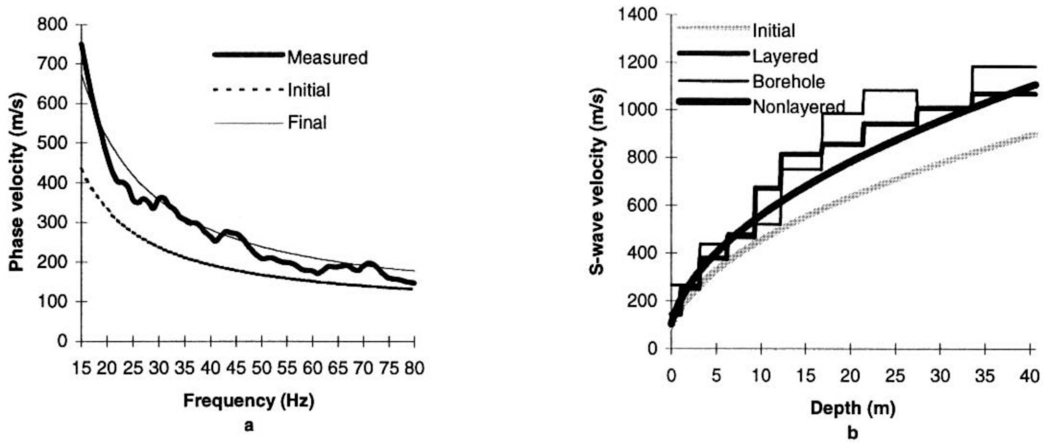

Fig. 1. Rayleigh wave dispersion curves (a) of the real example and S-wave velocity models (b).

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